

On the minimum rank of a graph over finite fields

Shmuel Friedland and Raphael Loewy

Department of Mathematics, Statistics, and Computer Science,
University of Illinois at Chicago
Chicago, Illinois 60607-7045, USA

Department of Mathematics
Technion – Israel Institute of Technology
32000 Haifa, Israel

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Abstract

In this paper we deal with two aspects of the minimum rank of a simple undirected graph G on n vertices over a finite field \mathbb{F}_q with q elements, which is denoted by $\text{mr}(\mathbb{F}_q, G)$. In the first part of this paper we show that the average minimum rank of simple undirected labeled graphs on n vertices over \mathbb{F}_2 is $(1 - \varepsilon_n)n$, where $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

In the second part of this paper we assume that G contains a clique K_k on k -vertices. We show that if q is not a prime then $\text{mr}(\mathbb{F}_q, G) \leq n - k + 1$ for $4 \leq k \leq n - 1$ and $n \geq 5$. It is known that $\text{mr}(\mathbb{F}_q, G) \leq 3$ for $k = n - 2$, $n \geq 4$ and $q \geq 4$. We show that for $k = n - 2$ and each $n \geq 10$ there exists a graph G such that $\text{mr}(\mathbb{F}_3, G) > 3$. For $k = n - 3$, $n \geq 5$ and $q \geq 4$ we show that $\text{mr}(\mathbb{F}_q, G) \leq 4$.

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1 Introduction.

Let $G = (V, E)$ be a simple undirected graph on the set of vertices V and the set of edges E . Set $n := |V|$, and identify V with $\langle n \rangle := \{1, \dots, n\}$. Denote by $ij \in E$ the edge connecting the vertices i and j . Let \mathbb{F} be a field. For a prime p let \mathbb{F}_p be the finite field of integers modulo p . Denote by $S(\mathbb{F}, G)$ the set of all symmetric $n \times n$ matrices $A = [a_{ij}]_{i,j=1}^n$ with entries in \mathbb{F} , and such that $a_{ij} \neq 0$, $i \neq j$, exactly when $ij \in E$. There is no restriction on the main diagonal entries of A . Let

$$\text{mr}(\mathbb{F}, G) = \min\{\text{rank } A : A \in S(\mathbb{F}, G)\}.$$

The problem of determining $\text{mr}(\mathbb{F}, G)$ has been of significant interest in recent years.

Here we consider two aspects of this problem. The first aspect is estimating the average of the minimum rank over \mathcal{G}_n , the set of all labeled graphs on $\langle n \rangle$. Since the complete graph on n vertices, denoted by $K_n = (\langle n \rangle, E_n)$ has $\binom{n}{2}$ edges, we have that $|\mathcal{G}_n| = 2^{\binom{n}{2}}$. Let

$$\alpha_n(\mathbb{F}) = \frac{1}{n2^{\binom{n}{2}}} \sum_{G \in \mathcal{G}_n} \text{mr}(\mathbb{F}, G) \quad (1.1)$$

be the *scaled* n -average minimum rank. It is a very interesting problem to estimate $\alpha_n(\mathbb{F})$ for large n , and

$$\bar{\alpha}(\mathbb{F}) := \limsup_{n \rightarrow \infty} \alpha_n(\mathbb{F}), \quad \underline{\alpha}(\mathbb{F}) := \liminf_{n \rightarrow \infty} \alpha_n(\mathbb{F}). \quad (1.2)$$

In this note we show

$$\underline{\alpha}(\mathbb{F}_2) = \bar{\alpha}(\mathbb{F}_2) = 1. \quad (1.3)$$

In a recent paper Hall, Hogben, Martin and Shader [HHMS] have shown that for n sufficiently large $0.146907 < \alpha_n(\mathbb{R}) < 0.5 + \frac{\sqrt{7 \ln n}}{n}$. The second aspect of this problem is estimating the minimum rank of graphs which contain a clique. A k -clique in G is a complete graph K_k that occurs as an induced subgraph of G .

We are interested in the following:

Question. Suppose G contains a k -clique. When is $\text{mr}(\mathbb{F}, G) \leq n - k + 1$?

To avoid trivialities, assume that $n = |V| \geq 3$. It is well known that the inequality $\text{mr}(\mathbb{F}, G) \leq n - 1$ always holds. It is also well known that $\text{mr}(\mathbb{F}, K_n) = 1$. Hence, our question has an affirmative answer for $k = 1, 2, n$. In fact, the same holds true for $k = 3$, by papers of Fiedler [F] and Bento-Leal Duarte [BD]. Hence we may assume from now on that

$$4 \leq k \leq n - 1.$$

The question has also an affirmative answer in case when \mathbb{F} is an infinite field. This appears implicitly in the paper of Johnson, Loewy and Smith [JLS]. (See §3.) In her M.Sc. thesis [B] Bank gave an affirmative answer to our question for any finite field \mathbb{F} with $|\mathbb{F}| \geq k - 1$. While giving us some information on the finite field case, this result has a drawback, namely, for a large k , the field is required to be large. However, as we will see here and in subsequent sections, at least for the special cases $k = n - 1$, $k = n - 2$ and $k = n - 3$ small fields suffice to get an affirmative answer to our question. Thus, the case of a finite field is still of significant interest. It follows from Barrett, van der Holst and Loewy [BvdHL] that our question has an affirmative answer in case $k = n - 1$ and \mathbb{F} is any field which is not \mathbb{F}_2 . As for \mathbb{F}_2 , let $n \geq 5$ and consider the graph G obtained from K_{n-1} by adding a new vertex v and connecting it to exactly two of the vertices of K_{n-1} . Then it follows from [BvdHL] that $\text{mr}(\mathbb{F}_2, G) = 3$. Thus, over \mathbb{F}_2 our question does not have an affirmative answer in all cases.

We now briefly survey the contents of this paper. In Section 2 we prove the equality (1.3); in Section 3 we deal briefly with the case of an infinite field; in Section 4 we deal with the case $k = n - 2$; in Section 5 we give an affirmative answer to our question for any finite field \mathbb{F} which is not a prime field; in Section 6 we consider the case $k = n - 3$.

2 Scaled average minimum rank over \mathbb{F}_2

We first recall some known results on certain classes of matrices $\mathbb{F}_2^{n \times n}$.

Lemma 2.1. *Let $O(n, \mathbb{F}_2)$ be the orthogonal group of $n \times n$ matrices. Then $O(n) := |O(n, \mathbb{F}_2)|$ is equal to $C(n)2^{\frac{n(n-1)}{2}}$, where*

$$C(1) = C(2) = 1, \text{ and } C(n) = \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (1 - 2^{-2i}) \text{ for } n > 2. \quad (2.1)$$

See for example [M, p. 158]. (Note that the formula in [M] has a different equivalent form.)

Denote by $S(n, \mathbb{F}_2) \subset \mathbb{F}_2^{n \times n}$ the subspace of symmetric matrices.

Lemma 2.2. *Let $A \in S(n, \mathbb{F}_2)$ have rank k . Then the A has the following form.*

1. *If k is odd then $A = XX^\top$, where $X \in \mathbb{F}_2^{n \times k}$ and $\text{rank } X = k$.*
2. *If $k = 2l$ is even then A has two possible nonequivalent forms. First $A = XX^\top$, where $X \in \mathbb{F}_2^{n \times k}$ and $\text{rank } X = k$. Second $A = X(\oplus_{j=1}^l H_2)X^\top$, where $X \in \mathbb{F}_2^{n \times k}$, $\text{rank } X = k$ and $H_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

See for example [Fr, Thm 2.6]. Let $J_{2n} \in S(2n, \mathbb{F}_2)$ be the direct sum of n copies of H_2 . Denote by $\text{Sym}(2n, \mathbb{F}_2) := \{T \in \mathbb{F}_2^{2n \times 2n}, T^\top J_{2n} T = J_{2n}\}$ the symplectic group of order $2n$ over \mathbb{F}_2 .

Lemma 2.3. *The cardinality of $\text{Sym}(2n, \mathbb{F}_2)$ is given by $O(2n + 1)$.*

See [C, p'6, p'11]. The following result is probably well known, and we bring its proof for completeness.

Lemma 2.4. *The number of $n \times k$ matrices $X \in \mathbb{F}_2^{n \times k}$ of rank $k \leq n$ is equal to*

$$N(n, k) = (2^n - 1)(2^n - 2) \dots (2^n - 2^{k-1}) = 2^{nk} \left(1 - \frac{1}{2^n}\right) \dots \left(1 - \frac{1}{2^{n-k+1}}\right) \quad (2.2)$$

Proof. The proof is by induction on k . For $k = 1$, A can have any first column, except the zero column. Hence $N(n, 1) = 2^n - 1$. Assume that the number of $n \times k$ matrices $X \in \mathbb{F}_2^{n \times k}$ of rank $k \leq n$ is equal to $N(n, k)$ for $k \leq n - 1$. Observe that $A \in \mathbb{F}_2^{n \times (k+1)}$ has rank $k + 1$ if and only if the first k columns of A are linearly independent, and the last column is not a linear combination of the first k columns. Assume that the first k columns of A are linearly independent. Then the number of vectors, which are linear combinations of the first k columns of A are 2^k . Hence the last column of A can be chosen in $2^n - 2^k$ ways such that $\text{rank } A = k + 1$. Thus $N(n, k + 1) = N(n, k)(2^n - 2^k)$. \square

Lemma 2.5. For $n \geq 2$

$$1 > \prod_{j=1}^{n-1} \left(1 - \frac{1}{2^j}\right) > \prod_{j=1}^{\infty} \left(1 - \frac{1}{2^j}\right) > \frac{1}{4}.$$

Proof. Recall that $\log(1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$ for $x \in (-1, 1)$.

$$\begin{aligned} \sum_{j=1}^{n-1} \log\left(1 - \frac{1}{2^j}\right) &> \sum_{j=1}^{\infty} \log\left(1 - \frac{1}{2^j}\right) = -\sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2^{jm}} = -\sum_{m=1}^{\infty} \frac{1}{m} \sum_{j=1}^{\infty} \frac{1}{2^{jm}} = \\ &= -\sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2^m(1 - 2^{-m})} > -\sum_{m=1}^{\infty} \frac{1}{m} \frac{2}{2^m} = 2 \log\left(1 - \frac{1}{2}\right) = \log \frac{1}{4}. \end{aligned}$$

\square

Combine Lemmas 2.1, 2.4 and 2.5 to deduce

$$1 \geq C(n) > \frac{1}{4}, \quad 2^{nk} > N(n, k) > 2^{nk-2}. \quad (2.3)$$

Lemma 2.6. Let $\theta(n, k)$ be the number of $A \in S(n, \mathbb{F}_2)$ of rank k . If k is odd then $\theta(n, k) = \frac{N(n, k)}{O(k)}$. If k even then $\theta(n, k) = \frac{N(n, k)}{O(k)} + \frac{N(n, k)}{O(k+1)}$. In particular

$$2^{(n - \frac{k-1}{2})k-2} < \theta(n, k) < 2^{(n - \frac{k-1}{2})k+3}. \quad (2.4)$$

Proof. We first find the number of distinct matrices $A \in S_n(\mathbb{F}_2)$ of rank k of the form XX^\top , where $X \in \mathbb{F}_2^{n \times k}$ of rank k . Assume that $X, Y \in \mathbb{F}_2^{n \times k}$ have both rank k and $A = XX^\top = YY^\top$. Since the columns of X and Y form a basis of the column space

of A , it follows that $Y = XQ$ for some $Q \in \mathbf{GL}(k, \mathbb{F}_2)$. We claim that $Q \in O(k, \mathbb{F}_2)$. Indeed, since the columns of X are linearly independent, they can be extended to a basis of \mathbb{F}_2^n . Hence, there exists $Z \in \mathbb{F}_2^{k \times n}$ such that $ZX = I_k$. So

$$I_k = Z(XX^\top)Z^\top = Z(YY^\top)Z^\top = QQ^\top.$$

Hence the number of such symmetric matrices of rank k is $\frac{N(n, k)}{O(k)}$. Use Lemmas 2.1, (2.1) and (2.3) to conclude that

$$2^{(n - \frac{k-1}{2})k-2} < \frac{N(n, k)}{O(k)} < 2^{(n - \frac{k-1}{2})k+2}.$$

If k is odd, we are done in view of Lemma 2.2. Suppose that k is even. Then Lemma 2.2 claims that we have a second kind of $A \in S(n, \mathbb{F}_2)$ of rank k , which is of the form XJ_kX^\top , where $X \in \mathbb{F}_2^{n \times k}$ is a matrix of rank k . As in the first case, $A = XJ_kX^\top = YJ_kY^\top$ if and only if $Y = XP$ where P is a symplectic matrix. In view of Lemma 2.3 the cardinality of the symplectic group over \mathbb{F}_2 of order k is $O(k+1)$. Hence the number the symmetric matrices of order n , rank k of the second kind is $\frac{N(n, k)}{O(k+1)}$, which is less than $\frac{N(n, k)}{O(k)}$. In particular (2.4) always holds. \square

Theorem 2.7. *Let $\alpha_n(\mathbb{F}_2)$ be the scaled n -average minimum rank over all simple graphs on n vertices over the field \mathbb{F}_2 , as defined by (1.1). Define $\underline{\alpha}(\mathbb{F}_2) \leq \overline{\alpha}(\mathbb{F}_2)$ as in (1.2). Then $\underline{\alpha}(\mathbb{F}_2) = \overline{\alpha}(\mathbb{F}_2) = 1$.*

Proof. Let us estimate the number of all graphs whose minimum rank is at most k . This number is at most the number of symmetric matrices in $S_n(\mathbb{F}_2)$ whose rank is at most k . (In other words, we assume the optimal condition that for each graph of minimum rank $r \leq k$ there is only one matrix of rank r and all other matrices are of rank greater than r .) Lemma 2.6 yields that the upper bound on this number is

$$\sum_{r=1}^k 8 \cdot 2^{(n - \frac{r-1}{2})r} < 8 \cdot 2^{(n - \frac{k-1}{2})k} \sum_{j=0}^{\infty} \frac{1}{2^j} = 16 \cdot 2^{(n - \frac{k-1}{2})k}.$$

The number of graphs is $2^{\frac{n(n-1)}{2}}$. Fix $t \in (0, 1)$. Suppose that $k \leq nt$. Then the number of all graphs with rank at most tn is less than $16 \cdot 2^{\frac{(2-t)tn^2}{2}}$. Note that

$$\lim_{n \rightarrow \infty} \frac{(tn)16 \cdot 2^{\frac{(2-t)tn^2}{2}}}{2^{\frac{n(n-1)}{2}}} = 0.$$

So the contribution of all these graphs to the average is zero. Thus all the contribution to $\underline{\alpha}(\mathbb{F}_2)$ comes from the graphs whose minimum rank is greater than tn for any $t \in (0, 1)$. Hence, $\underline{\alpha}(\mathbb{F}_2) \geq t$. Thus we showed (1.3). \square

3 Infinite fields

The question raised in the introduction has an affirmative answer in case \mathbb{F} is an infinite field. This appears implicitly in [JLS]. For the sake of clarity we state this result and give a short sketch of its proof.

Theorem A. *Let n and k be positive integers such that $4 \leq k < n$, and let G be a graph on n vertices which contains a k -clique as an induced subgraph. Then, for any infinite field F*

$$\text{mr}(\mathbb{F}, G) \leq n - k + 1.$$

Proof. We use the first part in the appendix of [JLS], and in particular Observation A.1, Lemma A.2 and the discussion between these two results.

We can assume that $1, 2, \dots, k$ are vertices of a k -clique in G . Given any i, j in $\{1, 2, \dots, k-1\}$ with $i \neq j$, there is a path of length two in G from i to j , namely the path whose only intermediate vertex is k . The conditions of Lemma A.2 in [JLS] are satisfied. Hence there exists a matrix $A \in S(\mathbb{F}, G)$ of the form

$$A = \begin{bmatrix} C & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix},$$

where A_{22} is an invertible $n - k + 1 \times n - k + 1$ matrix, and $C = A_{12}A_{22}^{-1}A_{12}^\top$. Then $\text{rank } A = n - k + 1$, implying $\text{mr}(\mathbb{F}, G) \leq n - k + 1$. \square

In light of Theorem A, we can assume from now on that F is a *finite field*.

4 The case $k = n - 2$.

In [B, Proposition 4.2.3] the following result has been proved.

Proposition 4.1. *Let G be a graph on $n \geq 4$ vertices and suppose that G contains K_{n-2} as an induced subgraph. Then $\text{mr}(\mathbb{F}, G) \leq 3$ for any field \mathbb{F} with $|\mathbb{F}| > 3$.*

It is known that Proposition 4.1 is not always valid when $\mathbb{F} = \mathbb{F}_2$. Examples of such graphs are given in [BGL], for example graph #14 there, whose minimum rank over \mathbb{F}_2 is 4.

The field \mathbb{F}_3 is not discussed in [B]. It is our purpose to show that Proposition 4.1 is not always valid when $\mathbb{F} = \mathbb{F}_3$, so in fact, it is best possible.

Theorem 4.2. *For every n such that $n \geq 10$ there exists a graph G on n vertices containing K_{n-2} as an induced subgraph, and such that $\text{mr}(\mathbb{F}_3, G) > 3$.*

Proof. Let G be a graph containing K_{n-2} as an induced subgraph. We label the vertices of G so that $1, 2, \dots, n-2$ are the vertices of an $(n-2)$ -clique of G . We assume that $n-2, n-1$ and n are independent vertices of G , that is no two of them are adjacent. Let $A \in S(\mathbb{F}_3, G)$ and partition A as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix},$$

where A_{22} is 3×3 . Note that each row vector of A_{12} is in \mathbb{F}_3^3 , and its first entry is nonzero, because vertex $n-2$ belongs to the given $(n-2)$ -clique. Applying, if necessary, a suitable congruence transformation $D^\top A D$, where D is an invertible diagonal matrix, we may assume without loss of generality that *the first entry* of each row vector of A_{12} is 1.

It follows that every row vector of A_{12} has one of the following four patterns,

- (i) $(1, 0, 0)$; (ii) $(1, *, 0)$; (iii) $(1, 0, *)$; (iv) $(1, *, *)$,

where $*$ denotes a nonzero element of \mathbb{F}_3 . We add now the following assumption on G (expressed in terms of A):

Assumption. *The matrix A_{12} has at least one row of the pattern (i), and at least two distinct rows of each of the other patterns.*

Observe first that each column of A_{12} is not a zero column. Suppose that $\text{mr}(\mathbb{F}_3, G) \leq 3$, and let $A \in \text{S}(\mathbb{F}_3, G)$ be such that $\text{rank } A = \text{mr}(\mathbb{F}_3, G)$. Consider A_{22} . The independence of vertices $n-2, n-1$ and n implies that $A_{22} = \text{diag}(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{F}_3$. We claim that A_{22} is invertible.

Indeed, A_{12} contains the row vectors $(1, *, *)$, $(1, 0, *)$ and $(1, 0, 0)$, and suppose that they are the i th, j th and k th rows of A_{12} . Since $\begin{bmatrix} 1 & * & * \\ 1 & 0 & * \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, it follows that $\text{rank } A \geq 3$, so we must have $\text{rank } A = 3$. Moreover, rows i, j and k of A span the row space of A . In particular, each of the nonzero rows $n-2, n-1$ and n of A is a linear combination of rows i, j , and k , and hence $\alpha, \beta, \gamma \neq 0$, so A_{22} is invertible.

Let $E = A_{22}$. Since $\text{rank } A = \text{rank } E = 3$, it follows, using Schur complement, that

$$A_{11} = A_{12}E^{-1}A_{12}^\top,$$

and, in particular, all off-diagonal entries of $A_{12}E^{-1}A_{12}^\top$ must be nonzero. We will assume from now on that $\alpha = 1$, as we can replace E by $2E$ if necessary, so that $E = \text{diag}(1, \beta, \gamma)$.

We will make repeated use of the requirement that all off-diagonal elements of $A_{12}E^{-1}A_{12}^\top$ must be nonzero. So if $x = (1, x_2, x_3)$ and $y = (1, y_2, y_3)$ are row vectors of any 2 distinct rows of A_{12} then

$$1 + \beta^{-1}x_2y_2 + \gamma^{-1}x_3y_3 = 1 + \beta x_2y_2 + \gamma x_3y_3 \neq 0. \quad (4.1)$$

Recall that $x^2 = 1$ for any $x \in \mathbb{F}_3 \setminus \{0\}$. Furthermore for $x, y \in \mathbb{F}_3 \setminus \{0\}$ $1 + xy \neq 0 \iff x = y$. We distinguish four cases.

Case 1. $\beta = \gamma = 1$, so $E = I_3$. Assume that $(1, x, 0), (1, 0, y)$ are two rows of the pattern (ii) and (iii) respectively. Let $(1, z_1, z_2)$ be a row of the pattern (iv). (4.1) for the pairs $(1, x, 0), \mathbf{z}$ and $(1, 0, y), \mathbf{z}$ yield that $\mathbf{z} = (1, x, y)$. Since there are at least two distinct rows of the form $(1, x, y)$ we contradict (4.1).

Case 2. $\beta = 1, \gamma = 2$, so $E = E^{-1} = \text{diag}(1, 1, 2)$.

Consider the rows of A_{12} with pattern $(1, 0, *)$. By (4.1), no two of them can be $(1, 0, 1)$ and no two of them can be $(1, 0, 2)$. Hence A_{12} has exactly two rows $(1, 0, 1)$ and $(1, 0, 2)$. Let $\mathbf{z} = (1, z_1, z_2)$ be a row of the pattern (iv). Then (4.1) cannot hold for the two pairs $(1, 0, 1), \mathbf{z}$ and $(1, 0, 2), \mathbf{z}$.

Case 3. $\beta = 2, \gamma = 1$, so $E = E^{-1} = \text{diag}(1, 2, 1)$.

The contradiction is obtained as in Case 2.

Case 4. $\beta = \gamma = 2$, so $E = E^{-1} = \text{diag}(1, 2, 2)$.

The contradiction is obtained as in Case 2. This concludes the proof of the theorem. \square

5 The case of a finite non-prime field.

In this section we assume that \mathbb{F} is any finite non-prime field. We prove

Theorem 5.1. *Let G be a graph on $n \geq 5$ vertices, and let $4 \leq k \leq n - 1$. Suppose that G contains K_k as an induced subgraph. Let \mathbb{F} be a finite non-prime field of characteristic p . Then $\text{mr}(\mathbb{F}, G) \leq n - k + 1$.*

Proof. The assumption on \mathbb{F} implies that \mathbb{F} is a finite extension of \mathbb{F}_p , and $\mathbb{F} \neq \mathbb{F}_p$. We label the vertices of G so that $1, 2, \dots, k$ are the vertices of a k -clique. Let $A \in \text{S}(\mathbb{F}, G)$ and partition A be as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix},$$

where A_{11} is $k \times k$. Let H be the subgraph of G induced by vertices $k+1, k+2, \dots, n$. It is straightforward to see that there exists $B \in \text{S}(\mathbb{F}_p, H)$ which is invertible. Indeed, let every nonzero off-diagonal entry of B be 1, and let $b_{11} = 1$. Then we can choose $b_{22} \in \mathbb{F}_p$ so that the principal minor of order 2 in the top left corner is 1. Similarly, we can sequentially choose $b_{33}, \dots, b_{n-k, n-k}$ in \mathbb{F}_p so that all leading principal minors of B are 1. We now pick every nonzero entry of A_{12} to be 1, and we let $A_{22} = \beta^{-1}B$, where $\beta \in \mathbb{F}/\mathbb{F}_p$. Hence $A_{22}^{-1} = \beta B^{-1}$, where all entries of B^{-1} are in \mathbb{F}_p .

We claim that A_{11} can be chosen so that $A_{11} \in S(\mathbb{F}, K_k)$ and so that

$$A_{11} - A_{12}A_{22}^{-1}A_{12}^\top = J_k,$$

where J_k is the all ones $k \times k$ matrix. Indeed, let

$$A_{11} = J_k + A_{12}A_{22}^{-1}A_{12}^\top = J_k + \beta A_{12}B^{-1}A_{12}^\top.$$

Every off-diagonal element of $J_k + \beta A_{12}B^{-1}A_{12}^\top$ is of the form $1 + \beta a$, $a \in \mathbb{F}_p$, and so must be nonzero. For our choice of A_{11}, A_{12} and A_{22} , the Schur complement of A_{22} in A has rank one, so $\text{rank } A = n - k + 1$. Hence $\text{mr}(\mathbb{F}, G) \leq n - k + 1$. \square

6 The case $k = n - 3$.

In this section we consider the case $k = n - 3$, that is, we assume that G contains an $(n - 3)$ -clique. We prove

Theorem 6.1. *Let G be a graph on $n \geq 5$ vertices, and suppose that G contains K_{n-3} as an induced subgraph. Then, $\text{mr}(\mathbb{F}, G) \leq 4$ for every field \mathbb{F} with $|\mathbb{F}| > 3$.*

Proof. We assume that \mathbb{F} is a finite field with $|\mathbb{F}| > 3$. Let us label the vertices of G so that $1, 2, \dots, n - 3$ are the vertices of an $(n - 3)$ -clique in G . Let $A \in S(\mathbb{F}, G)$ and partition A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix},$$

where A_{22} is 3×3 . Our goal is to show that A can be chosen so that A_{22} is invertible and its Schur complement in A is J_{n-3} . Then it follows for this A that $\text{rank } A = 4$, implying $\text{mr}(\mathbb{F}, G) \leq 4$.

Denote by $\mathbf{1}$ the all ones vector (of order that should be clear from the discussion). Each column vector of A_{12}^\top must be of one of the following eight patterns:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ * \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}, \begin{bmatrix} 0 \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix},$$

where $*$ denotes a nonzero entry. We may assume without loss of generality that the columns of A_{12}^\top are already arranged in eight groups (although not all of them must be present) according to these patterns. Moreover, we pick the nonzero entries of A_{12}^\top so that all columns with the same pattern are equal. In fact, we pick the entries of A_{12}^\top so that

$$A_{12}^\top = \begin{bmatrix} 0 & \alpha_1 \mathbf{1}^\top & 0 & 0 & \alpha_4 \mathbf{1}^\top & \alpha_6 \mathbf{1}^\top & 0 & \alpha_{10} \mathbf{1}^\top \\ 0 & 0 & \alpha_2 \mathbf{1}^\top & 0 & \alpha_5 \mathbf{1}^\top & 0 & \alpha_8 \mathbf{1}^\top & \alpha_{11} \mathbf{1}^\top \\ 0 & 0 & 0 & \alpha_3 \mathbf{1}^\top & 0 & \alpha_7 \mathbf{1}^\top & \alpha_9 \mathbf{1}^\top & \alpha_{12} \mathbf{1}^\top \end{bmatrix} \in \mathbb{F}^{3, n-3},$$

where $\alpha_1, \alpha_2, \dots, \alpha_{12}$ will be determined later, and not all block columns must be present.

We will distinguish four cases, according to the pattern of the entries of A_{22} . In each case we pick the entries of A_{12} and A_{22} so that A_{22} is nonsingular and so that all off-diagonal entries of $J_{n-3} + A_{12}A_{22}^{-1}A_{12}^\top$ are nonzero. Then we let $A_{11} = J_{n-3} + A_{12}A_{22}^{-1}A_{12}^\top$, and we get what we want.

Case 1. Suppose that A_{22} is a diagonal matrix. we let $A_{22} = \beta^{-1}I_3$, where $0 \neq \beta \in \mathbb{F}$. Hence $A_{22}^{-1} = \beta I_3$. A straightforward computation yields

$$A_{12}A_{22}^{-1} = \beta \begin{bmatrix} 0 & 0 & 0 \\ \alpha_1 \mathbf{1} & 0 & 0 \\ 0 & \alpha_2 \mathbf{1} & 0 \\ 0 & 0 & \alpha_3 \mathbf{1} \\ \alpha_4 \mathbf{1} & \alpha_5 \mathbf{1} & 0 \\ \alpha_6 \mathbf{1} & 0 & \alpha_7 \mathbf{1} \\ 0 & \alpha_8 \mathbf{1} & \alpha_9 \mathbf{1} \\ \alpha_{10} \mathbf{1} & \alpha_{11} \mathbf{1} & \alpha_{12} \mathbf{1} \end{bmatrix},$$

so all off-diagonal entries of $A_{12}A_{22}^{-1}A_{12}^\top$ are contained in

$$\begin{aligned} & \left\{ 0, \beta\alpha_1^2, \beta\alpha_1\alpha_4, \beta\alpha_1\alpha_6, \beta\alpha_1\alpha_{10}, \beta\alpha_2^2, \beta\alpha_2\alpha_5, \beta\alpha_2\alpha_8, \beta\alpha_2\alpha_{11}, \beta\alpha_3^2, \beta\alpha_3\alpha_7, \right. \\ & \beta\alpha_3\alpha_9, \beta\alpha_3\alpha_{12}, \beta(\alpha_4^2 + \alpha_5^2), \beta\alpha_4\alpha_6, \beta\alpha_5\alpha_8, \beta(\alpha_4\alpha_{10} + \alpha_5\alpha_{11}), \\ & \beta(\alpha_6^2 + \alpha_7^2), \beta\alpha_7\alpha_9, \beta(\alpha_6\alpha_{10} + \alpha_7\alpha_{12}), \beta(\alpha_8^2 + \alpha_9^2), \beta(\alpha_8\alpha_{11} + \alpha_9\alpha_{12}), \\ & \left. \beta(\alpha_{10}^2 + \alpha_{11}^2 + \alpha_{12}^2) \right\}. \end{aligned}$$

Choose now all α 's to be 1. So the constraints on β are

$$\beta \neq 0; 1 + \beta \neq 0; 1 + 2\beta \neq 0; 1 + 3\beta \neq 0.$$

For any \mathbb{F} with $|\mathbb{F}| \geq 5$ one can choose $\beta \in F$ to satisfy these constraints. For \mathbb{F} with $|\mathbb{F}| = 4$ one can apply Theorem 5.1 or note that for this field $2 = 0$ and $3 = 1$, so there are only two constraints.

Case 2. Suppose that exactly two of the entries above the main diagonal of A_{22} are zero. We assume that they are the elements in the 1, 2 and 1, 3 positions (other possibilities are handled in a similar way). Let

$$A_{22} = \beta^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ where } 0 \neq \beta \in \mathbb{F}, \text{ so } A_{22}^{-1} = \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence

$$A_{12}A_{22}^{-1} = \beta \begin{bmatrix} 0 & 0 & 0 \\ \alpha_1 \mathbf{1} & 0 & 0 \\ 0 & 0 & \alpha_2 \mathbf{1} \\ 0 & \alpha_3 \mathbf{1} & 0 \\ \alpha_4 \mathbf{1} & 0 & \alpha_5 \mathbf{1} \\ \alpha_6 \mathbf{1} & \alpha_7 \mathbf{1} & 0 \\ 0 & \alpha_9 \mathbf{1} & \alpha_8 \mathbf{1} \\ \alpha_{10} \mathbf{1} & \alpha_{12} \mathbf{1} & \alpha_{11} \mathbf{1} \end{bmatrix},$$

and a discussion similar to the one in Case 1 shows that if all α 's are chosen to be 1 then the constraints on β are again

$$\beta \neq 0; 1 + \beta \neq 0; 1 + 2\beta \neq 0; 1 + 3\beta \neq 0.$$

Case 3. Suppose that exactly one entry above the main diagonal of A_{22} is zero. We assume that it is the 1, 2 entry (other possibilities are handled similarly). Let $A_{22} = \beta^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, where $0 \neq \beta \in \mathbb{F}$, so $A_{22}^{-1} = \beta \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Hence

$$A_{12}A_{22}^{-1} = \beta \begin{bmatrix} 0 & 0 & 0 \\ \alpha_1 \mathbf{1} & -\alpha_1 \mathbf{1} & \alpha_1 \mathbf{1} \\ -\alpha_2 \mathbf{1} & \alpha_2 \mathbf{1} & 0 \\ \alpha_3 \mathbf{1} & 0 & 0 \\ (\alpha_4 - \alpha_5) \mathbf{1} & (-\alpha_4 + \alpha_5) \mathbf{1} & \alpha_4 \mathbf{1} \\ (\alpha_6 + \alpha_7) \mathbf{1} & -\alpha_6 \mathbf{1} & \alpha_6 \mathbf{1} \\ (-\alpha_8 + \alpha_9) \mathbf{1} & \alpha_8 \mathbf{1} & 0 \\ (\alpha_{10} - \alpha_{11} + \alpha_{12}) \mathbf{1} & (-\alpha_{10} + \alpha_{11}) \mathbf{1} & \alpha_{10} \mathbf{1} \end{bmatrix},$$

so all of the diagonal entries of $A_{12}A_{22}^{-1}A_{12}^\top$ are contained in

$$\begin{aligned} & \left\{ 0, \beta\alpha_1^2, -\beta\alpha_1\alpha_2, \beta\alpha_1\alpha_3, \beta\alpha_1(\alpha_4 - \alpha_5), \beta\alpha_1(\alpha_6 + \alpha_7), \beta\alpha_1(-\alpha_8 + \alpha_9), \right. \\ & \beta\alpha_1(\alpha_{10} - \alpha_{11} + \alpha_{12}), \beta\alpha_2^2, \beta\alpha_2(-\alpha_4 + \alpha_5), -\beta\alpha_2\alpha_6, \beta\alpha_2\alpha_8, \beta\alpha_2(-\alpha_{10} + \alpha_{11}), \\ & \beta\alpha_3\alpha_4, \beta\alpha_3\alpha_6, \beta\alpha_3\alpha_{10}, \beta(\alpha_4(\alpha_4 - \alpha_5) + \alpha_5(-\alpha_4 + \alpha_5)), \beta(\alpha_6(\alpha_4 - \alpha_5) + \alpha_4\alpha_7), \\ & \beta(\alpha_8(-\alpha_4 + \alpha_5) + \alpha_4\alpha_9), \beta(\alpha_{10}(\alpha_4 - \alpha_5) + \alpha_{11}(-\alpha_4 + \alpha_5) + \alpha_4\alpha_{12}), \\ & \beta(\alpha_6(\alpha_6 + \alpha_7) + \alpha_6\alpha_7), \beta(-\alpha_6\alpha_8 + \alpha_6\alpha_9), \beta(\alpha_{10}(\alpha_6 + \alpha_7) - \alpha_6\alpha_{11} + \alpha_6\alpha_{12}), \\ & \beta\alpha_8^2, \beta(\alpha_{10}(-\alpha_8 + \alpha_9) + \alpha_8\alpha_{11}), \\ & \left. \beta(\alpha_{10}(\alpha_{10} - \alpha_{11} + \alpha_{12}) + \alpha_{11}(-\alpha_{10} + \alpha_{11}) + \alpha_{10}\alpha_{12}) \right\}. \end{aligned}$$

We can assume that $|\mathbb{F}| \neq 4$, as our theorem holds for the field with four elements by Theorem 5.1. Suppose also that $\mathbb{F} \neq \mathbb{F}_5$, so $|\mathbb{F}| > 5$. Let all α 's be chosen to be 1. Then the constraints on β are

$$\beta \neq 0; \ 1 + \beta \neq 0; \ 1 - \beta \neq 0; \ 1 + 2\beta \neq 0; \ 1 + 3\beta \neq 0,$$

and they can be satisfied. It remains to consider the case $\mathbb{F} = \mathbb{F}_5$. In this case we let $\alpha_7 = -1$ and all other α 's be 1. A straightforward computation shows that the constraints on β are now

$$\beta \neq 0; \ 1 + \beta \neq 0; \ 1 - \beta \neq 0; \ 1 + 2\beta \neq 0,$$

so $\beta = 3$ works.

Case 4. We assume now that all off-diagonal entries of A_{22} are nonzero. We pick $a \in \mathbb{F}$ so that $a \neq 0$. By Theorem 5.1, we may assume that the characteristic of $\mathbb{F} \neq 2$. Let

$$A_{22} = \frac{1}{2a} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Then all off-diagonal entries of A_{22} are nonzero, A_{22} is invertible, and

$$A_{22}^{-1} = \begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{bmatrix}.$$

Hence,

$$A_{12}A_{22}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 a \mathbf{1} & \alpha_1 a \mathbf{1} \\ \alpha_2 a \mathbf{1} & 0 & \alpha_2 a \mathbf{1} \\ \alpha_3 a \mathbf{1} & \alpha_3 a \mathbf{1} & 0 \\ \alpha_5 a \mathbf{1} & \alpha_4 a \mathbf{1} & (\alpha_4 + \alpha_5) a \mathbf{1} \\ \alpha_7 a \mathbf{1} & (\alpha_6 + \alpha_7) a \mathbf{1} & \alpha_6 a \mathbf{1} \\ (\alpha_8 + \alpha_9) a \mathbf{1} & \alpha_9 a \mathbf{1} & \alpha_8 a \mathbf{1} \\ (\alpha_{11} + \alpha_{12}) a \mathbf{1} & (\alpha_{10} + \alpha_{12}) a \mathbf{1} & (\alpha_{10} + \alpha_{11}) a \mathbf{1} \end{bmatrix},$$

so all off-diagonal entries of $A_{12}A_{22}^{-1}A_{12}^\top$ are contained in

$$\left\{ 0, \alpha_1 \alpha_2 a, \alpha_1 \alpha_3 a, \alpha_1 \alpha_5 a, \alpha_1 \alpha_7 a, \alpha_1 (\alpha_8 + \alpha_9) a, \alpha_1 (\alpha_{11} + \alpha_{12}) a, \right. \\ \alpha_2 \alpha_3 a, \alpha_2 \alpha_4 a, \alpha_2 (\alpha_6 + \alpha_7) a, \alpha_2 \alpha_9 a, \alpha_2 (\alpha_{10} + \alpha_{12}) a, \alpha_3 (\alpha_4 + \alpha_5) a, \alpha_3 \alpha_6 a, \\ \alpha_3 \alpha_8 a, \alpha_3 (\alpha_{10} + \alpha_{11}) a, 2\alpha_4 \alpha_5 a, (\alpha_5 \alpha_6 + (\alpha_4 + \alpha_5) \alpha_7) a, (\alpha_4 \alpha_8 + (\alpha_4 + \alpha_5) \alpha_9) a, \\ (\alpha_5 \alpha_{10} + \alpha_4 \alpha_{11} + (\alpha_4 + \alpha_5) \alpha_{12}) a, 2\alpha_6 \alpha_7 a, ((\alpha_6 + \alpha_7) \alpha_8 + \alpha_6 \alpha_9) a, \\ (\alpha_7 \alpha_{10} + (\alpha_6 + \alpha_7) \alpha_{11} + \alpha_6 \alpha_{12}) a, 2\alpha_8 \alpha_9 a, ((\alpha_8 + \alpha_9) \alpha_{10} + \alpha_9 \alpha_{11} + \alpha_8 \alpha_{12}) a, \\ \left. ((\alpha_{11} + \alpha_{12}) \alpha_{10} + (\alpha_{10} + \alpha_{12}) \alpha_{11} + (\alpha_{10} + \alpha_{11}) \alpha_{12}) a \right\}.$$

Suppose first that $\mathbb{F} \neq \mathbb{F}_5$, so $|\mathbb{F}| \geq 7$. Letting all α 's be 1 we obtain the constraints on a :

$$a \neq 0; 1 + a \neq 0; 1 + 2a \neq 0; 1 + 3a \neq 0; 1 + 4a \neq 0; 1 + 6a \neq 0.$$

These constraints can be satisfied in any finite field with at least seven elements.

In case $\mathbb{F} = \mathbb{F}_5$, let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_6 = \alpha_8 = \alpha_{10} = 1$, $\alpha_5 = \alpha_7 = \alpha_9 = \alpha_{11} = -1$ and $\alpha_{12} = 2$. This choice yields the constraints

$$a \neq 0, 1 + a \neq 0; 1 - a \neq 0; 1 + 3a \neq 0,$$

so the choice $a = 2$ works. □

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